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More non-Abelian loop Toda solitons

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Abstract

We find new solutions, including soliton-like ones, for a special case of non-Abelian loop Toda equations associated with complex general linear groups. We use the method of rational dressing based on an appropriate block-matrix representation suggested by the $\mathbb{Z}$-gradation under consideration. We present solutions in the form of a direct matrix generalization of Hirota's soliton solution already well known in the case of Abelian loop Toda systems.

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1. Introduction

The two-dimensional Toda equations play an essential role in understanding certain structures in classical and quantum integrable systems. They are formulated as nonlinear partial differential equations of second order and are associated with Lie groups, see, for example, the monographs [1, 2]. The Toda equations associated with affine Kac–Moody groups are of special interest, because they possess soliton solutions having a lot of physical applications. The simplest example here is the celebrated sine-Gordon equation known for a long time. Another example of an affine Toda equation was constructed in the paper [3] as a direct two-dimensional generalization of the famous mechanical Toda chain. Later on, the consideration of the paper [3] was generalized in the papers [4, 5] in the case of Toda chains related to various affine Kac–Moody algebras. Another approach to formulating affine Toda systems, based on folding properties of Dynkin diagrams, was implemented in [6]. Note also that the papers [3, 4, 5, 7] were pioneering in investigating the question of integrability of affine Toda field theories by considering the corresponding zero-curvature representation.

It is convenient to consider instead of the Toda systems associated with affine Kac–Moody groups the Toda systems associated with loop groups. There are two reasons to do so. First, the affine Kac–Moody groups can be considered as a loop extension of loop groups and

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the solutions of the corresponding Toda equations are connected in a simple way, see, for example, the paper [8]. Second, in distinction to the loop groups, there is no realization of affine Kac–Moody groups suitable for practical usage.

It is known that a Toda equation associated with a Lie group is specified by the choice of a $\mathbb{Z}$-gradation of its Lie algebra [1, 2]. Hence, to classify the Toda systems associated with some class of Lie groups one needs to describe all $\mathbb{Z}$-gradations of the respective Lie algebras. Recently, in a series of papers [9–11], we classified a wide class of Toda equations associated with untwisted and twisted loop groups of complex classical Lie groups. More concretely, we introduced the notion of an integrable $\mathbb{Z}$-gradation of a loop Lie algebra, and found all such gradations with finite-dimensional grading subspaces for the loop Lie algebras of complex classical Lie algebras. Then we described the respective Toda equations. It appeared that despite the fact that we consider Toda equations associated with infinite-dimensional Lie groups, the resulting Toda equations are equivalent to the equations formulated only in terms of the underlying finite-dimensional Lie groups and Lie algebras. Actually, partial cases of such types of equations appeared before, see, for example, the papers [12–14] and references therein, but we demonstrated that any Toda equation of the class under consideration can be written in terms of finite-dimensional Lie groups and Lie algebras. To slightly simplify the terminology and make a distinction from the Toda equations associated with finite-dimensional Lie groups, we call the finite-dimensional version of a Toda equation associated with a loop group of the Lie group G a loop Toda equation associated with the Lie group G.

Here we consider untwisted loop Toda equations associated with the complex general linear group. As was shown in the papers [10, 11], any such equation has the form[3]

$$\partial_+(\gamma^{-1}\partial_-\gamma) = [c_-, \gamma^{-1}c_+\gamma], \tag{1}$$

supplied with the conditions

$$\partial_+ c_- = 0, \qquad \partial_- c_+ = 0. \tag{2}$$

Here, γ is a mapping of the two-dimensional manifold $\mathcal{M}$ to the complex general linear group $\mathrm{GL}_n(\mathbb{C})$ having a block-diagonal form

$$\gamma = \begin{pmatrix} \Gamma_1 & & & \\ & \Gamma_2 & & \\ & & \ddots & \\ & & & \Gamma_p \end{pmatrix},$$

so that for each $\alpha = 1, \ldots, p$ the mapping Γ_α is a mapping of $\mathcal{M}$ to the Lie group $\mathrm{GL}_{n_\alpha}(\mathbb{C})$ with $\sum_{\alpha=1}^{p} n_\alpha = n$. Further, c_+ and c_- are mappings of $\mathcal{M}$ to the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$. The mapping c_+ has the block-matrix structure

$$c_+ = \begin{pmatrix} 0 & C_{+1} & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & C_{+(p-1)} \\ C_{+0} & & & & 0 \end{pmatrix},$$

[3] We denote by ∂_+ and ∂_- the partial derivatives over the standard coordinates z^+ and z^- of a smooth two-dimensional manifold $\mathcal{M}$, where $\mathcal{M}$ is either the Euclidean plane $\mathbb{R}^2$ or the complex line $\mathbb{C}$; in the latter case z^- denotes the standard complex coordinate on $\mathbb{C}$, and z^+ – its complex conjugate.

where for each $\alpha = 1, \ldots, p-1$ the mapping $C_{+\alpha}$ is a mapping of $\mathcal{M}$ to the space of $n_\alpha \times n_{\alpha+1}$ complex matrices, and C_{+0} is a mapping of $\mathcal{M}$ to the space of $n_p \times n_1$ complex matrices. The mapping c_- has a similar block-matrix structure,

$$c_- = \begin{pmatrix} 0 & & & & C_{-0} \\ C_{-0} & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 0 & \\ & & & C_{-(p-1)} & 0 \end{pmatrix},$$

where for each $\alpha = 1, \ldots, p-1$ the mapping $C_{-\alpha}$ is a mapping of $\mathcal{M}$ to the space of $n_{\alpha+1} \times n_\alpha$ complex matrices, and C_{-0} is a mapping of $\mathcal{M}$ to the space of $n_1 \times n_p$ complex matrices. It is assumed that the mappings c_+ and c_- are fixed, and equation (1) is considered as an equation for the mapping γ, which can be written explicitly as a system of equations for the mappings Γ_α.

It is worth noting that for arbitrary complex classical Lie groups loop Toda equations belonging to the class under consideration have the same form (1) with the same block-matrix structure of the mappings γ, c_+ and c_-, but with some additional restrictions imposed on the blocks. Note also that the Toda equation under consideration is Abelian if the mapping γ is effectively a mapping to an Abelian Lie group, otherwise we have a non-Abelian Toda equation.

In the present paper we consider a particular case of non-Abelian loop Toda equations associated with the complex general linear group $\mathrm{GL}_n(\mathbb{C})$, where $n_\alpha = n/p = n_*$ for all $\alpha = 1, \ldots, p$. Moreover, we assume for simplicity that all nonzero entries of the block-matrix representation of c_+ and c_- are unit $n_* \times n_*$ matrices. In this case the Toda equation (1) can be written as an infinite periodic system,

$$\partial_+\left(\Gamma_\alpha^{-1}\partial_-\Gamma_\alpha\right) + \Gamma_\alpha^{-1}\Gamma_{\alpha+1} - \Gamma_{\alpha-1}^{-1}\Gamma_\alpha = 0, \tag{3}$$

with Γ_α subject to the condition $\Gamma_{\alpha+p} = \Gamma_\alpha$. This particular case of Toda systems was introduced in the remarkable paper by Mikhailov [4].

Here we are interested in explicit solutions of the system (3), in particular, in the soliton-like ones in the non-Abelian case when $n_* > 1$. Soliton solutions of the Abelian loop Toda equations can be found by various methods. The most known and elaborated among them are the Hirota's method [15], successfully applied to many particular cases of Abelian affine Toda systems [8, 16–20]; the vertex operators approach of [21–23] based on a proper specialization of the Leznov–Saveliev method [24], see also [25, 26] for more details and the relation to the dressing symmetry; and the formalism of rational dressing developed by Mikhailov [4] on the basis of a general dressing procedure proposed by Zakharov and Shabat [27]. Actually, in all known cases of Abelian Toda systems the vertex operators constructions reproduce the same soliton solutions found by Hirota's approach. In the paper [28] we considered Abelian untwisted loop Toda equations associated with complex general linear groups within the frameworks of Hirota's and rational dressing methods and established the explicit relationships between solutions given by these two approaches. Further, in the paper [29], using the rational dressing method, we have constructed multi-soliton solutions for Abelian twisted loop Toda systems associated with general linear groups.

There are not many papers dealing with soliton solutions of non-Abelian loop Toda equations. We would like to mention here the paper [30] where a combination of the notion of a quasi-determinant and the Marchenko lemmas were used to construct soliton-like solutions of equations (3). In the paper [31] we have developed the rational dressing method in application to non-Abelian untwisted loop Toda equations associated with complex general linear groups and found certain multi-soliton solutions. Here we restrict our attention to equations (3) and show that the rational dressing method in this most symmetric case allows one to construct new soliton-like solutions which can be presented in the form of a direct matrix generalization of Hirota's soliton solutions well known in the case of Abelian loop Toda systems.

The main idea of this paper, as of the paper [31], is to demonstrate the power of the rational dressing formalism appropriately developed for explicitly constructing solutions to the non-Abelian loop Toda equations. It is a distinctive feature of this method that it allows for such remarkable generalizations to the non-Abelian case, where other well-known methods fail to work. Among the solutions to be presented in what follows, we single out a class of soliton-like ones thus justifying the title of our paper. Here, by an n-soliton, or soliton-like, solution we mean a solution depending on n linear combinations of independent variables and having an appropriate number of characteristic parameters.

2. Rational dressing

We see that the constant matrices c_- and c_+ commute. Hence, it is obvious that

$$\gamma = I_n, \tag{4}$$

where I_n is the $n \times n$ unit matrix, is a solution to the Toda equation (1). For the formalism of rational dressing it is crucial that the Toda equation (1), (2) can be represented as the zero-curvature condition for a flat connection in a trivial principal fiber bundle, satisfying the grading and gauge-fixing conditions, see, for example, the books [1, 2]. Actually, having such a connection we have a solution to the Toda equations (1). In our case, the corresponding base manifold $\mathcal{M}$ is either the Euclidean plane $\mathbb{R}^2$ or the complex manifold $\mathbb{C}$, and the fiber coincides with the untwisted loop group $\mathcal{L}_{a,p}(\mathrm{GL}_n(\mathbb{C}))$, where a denotes the inner automorphism of $\mathrm{GL}_n(\mathbb{C})$ of order p acting on an element $g \in \mathrm{GL}_n(\mathbb{C})$ in accordance with the equality

$$a(g) = hgh^{-1}.$$

Here h is a block-diagonal matrix defined by the relation

$$h_{\alpha\beta} = \epsilon_p^{p-\alpha+1} I_{n_*} \delta_{\alpha\beta}, \qquad \alpha, \beta = 1, \ldots, p, \tag{5}$$

where $\epsilon_p = \mathrm{e}^{2\pi \mathrm{i}/p}$ is the pth principal root of unity.

Using the exponential law [32, 33], it is convenient to identify the mapping generating a flat connection under consideration with a smooth mapping of $\mathcal{M} \times S^1$ to $\mathrm{GL}_n(\mathbb{C})$ and the connection components with smooth mappings of $\mathcal{M} \times S^1$ to $\mathfrak{gl}_n(\mathbb{C})$. Below we think of the circle S^1 as consisting of complex numbers of modulus one.

Denote the mapping generating the connection corresponding to the solution (4) by φ. For the case under consideration, the rational dressing method consists in finding a mapping ψ of $\mathcal{M} \times S^1$ to $\mathrm{GL}_n(\mathbb{C})$, such that the grading and gauge-fixing conditions for the components of the flat connection generated by the mapping $\varphi\psi$ are satisfied. We assume that the analytic extension of the mapping ψ from S^1 to the whole Riemann sphere is a rational mapping given by the expression

$$\psi = \left(I_n + \sum_{i=1}^{r} \sum_{k=1}^{p} \frac{\lambda}{\lambda - \epsilon_p^k \mu_i} h^k P_i h^{-k} \right) \psi_0. \tag{6}$$

Here λ is the standard coordinate in $\mathbb{C}$, ψ_0 is a mapping of $\mathcal{M}$ to the Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$ formed by the elements $g \in \mathrm{GL}_n(\mathbb{C})$ subject to the equality $hgh^{-1} = g$, and P_i are some smooth mappings of $\mathcal{M}$ to the algebra $\mathrm{Mat}_n(\mathbb{C})$ of $n \times n$ complex matrices. It is also assumed here that $\mu_i \neq 0$, $\mu_i^p \neq \mu_j^p$ for all $i \neq j$. Expression (6) is obtained from an initial rational mapping by averaging over the action of the inner automorphism a, where it was used that $a^p = \mathrm{id}_{\mathrm{GL}(\mathbb{C})}$. Further, we suppose that the analytic extension of the corresponding inverse mapping to the whole Riemann sphere has a similar structure,

$$\psi^{-1} = \psi_0^{-1}\left(I_n + \sum_{i=1}^{r}\sum_{k=1}^{p} \frac{\lambda}{\lambda - \epsilon_p^k \nu_i} h^k Q_i h^{-k}\right),$$

with the pole positions satisfying the conditions $\nu_i \neq 0$, $\nu_i^p \neq \nu_j^p$ for all $i \neq j$, and additionally $\nu_i^p \neq \mu_j^p$ for any i and j. Note that the mappings $\psi(\lambda)$ and $\psi^{-1}(\lambda)$ are regular at the points $\lambda = 0$ and $\lambda = \infty$.

By definition, the equality

$$\psi^{-1}\psi = I_n$$

is valid on S^1. Since ψ and ψ^{-1} are rational mappings, this equality is valid on the whole Riemann sphere. Therefore, the residues of $\psi^{-1}\psi$ at the points μ_i and ν_i must vanish. This leads to certain relations to be satisfied by the mappings P_i and Q_i,

$$Q_i\left(I_n + \sum_{j=1}^{r}\sum_{k=1}^{p} \frac{\nu_i}{\nu_i - \epsilon_p^k \mu_j} h^k P_j h^{-k}\right) = 0, \tag{7}$$

$$\left(I_n + \sum_{j=1}^{r}\sum_{k=1}^{p} \frac{\mu_i}{\mu_i - \epsilon_p^k \nu_j} h^k Q_j h^{-k}\right) P_i = 0. \tag{8}$$

Further, for the components of the flat connection generated by the mapping $\varphi\psi$ we find the expressions

$$\omega_- = \psi^{-1}\partial_-\psi + \lambda^{-1}\psi^{-1}c_-\psi,$$
$$\omega_+ = \psi^{-1}\partial_+\psi + \lambda\psi^{-1}c_+\psi.$$

We see that ω_- is a rational mapping having simple poles at μ_i, ν_i and zero. Similarly, ω_+ is a rational mapping having simple poles at μ_i, ν_i and infinity. We need a connection satisfying the grading and gauge-fixing conditions. The grading condition in our case means that for each point of $\mathcal{M}$ the component $\omega_-(\lambda)$ is rational and has the only simple pole at zero, and the component $\omega_+(\lambda)$ is rational and has the only simple pole at infinity. Therefore, we require that the residues of ω_- and ω_+ at the points μ_i and ν_i must vanish. And this requirement imposes additional conditions on the mappings P_i and Q_i, that are

$$\left(\partial_- Q_i - \nu_i^{-1} Q_i c_-\right)\left(I_n + \sum_{j=1}^{r}\sum_{k=1}^{p} \frac{\nu_i}{\nu_i - \epsilon_p^k \mu_j} h^k P_j h^{-k}\right) = 0, \tag{9}$$

$$\left(\partial_+ Q_i - \nu_i Q_i c_+\right)\left(I_n + \sum_{j=1}^{r}\sum_{k=1}^{p} \frac{\nu_i}{\nu_i - \epsilon_p^k \mu_j} h^k P_j h^{-k}\right) = 0 \tag{10}$$

for the residues at the points ν_i, and also

$$\left(I_n + \sum_{j=1}^{r}\sum_{k=1}^{p} \frac{\mu_i}{\mu_i - \epsilon_p^k \nu_j} h^k Q_j h^{-k}\right)\left(\partial_- P_i + \mu_i^{-1} c_- P_i\right) = 0, \tag{11}$$

$$\left(I_n + \sum_{j=1}^{r}\sum_{k=1}^{p} \frac{\mu_i}{\mu_i - \epsilon_p^k \nu_j} h^k Q_j h^{-k}\right)(\partial_+ P_i + \mu_i c_+ P_i) = 0 \tag{12}$$

for the residues at the points μ_i. Having these and the above-mentioned relations fulfilled by P_i and Q_i, and besides assuming that $\psi_0 = I_n$ that resolves the gauge-fixing constraint $\omega_{+0} = 0$, where we put $\omega_{+0} = \omega_+(0)$, we can see that the mapping $\gamma = \psi(\infty)$, with fixed $2r$ complex numbers μ_i, ν_i, satisfies the Toda equation (1). In the block-matrix form we have the expressions

$$\gamma_{\alpha\beta} = \delta_{\alpha\beta}\left(I_{n_*} + p\sum_{i=1}^{r}(P_i)_{\alpha\alpha}\right), \qquad \gamma^{-1}_{\alpha\beta} = \delta_{\alpha\beta}\left(I_{n_*} + p\sum_{i=1}^{r}(Q_i)_{\alpha\alpha}\right), \tag{13}$$

where $(P_i)_{\alpha\beta}$ and $(Q_i)_{\alpha\beta}$ are $n_* \times n_*$ complex matrices satisfying certain conditions (7)–(12). These conditions ensuring the vanishing of the residues of $\psi^{-1}\psi$, ω_- and ω_+ at the points ν_i, μ_i can be non-trivially fulfilled, see also the papers [28, 31].

To make the solution (13) explicit we should specify the matrix-valued functions P_i and Q_i. We first note that, if we suppose that the functions P_i and Q_i take values in the space of the matrices of maximum rank, then we come to the trivial solution (4). Hence, we assume that the values of the functions P_i and Q_i are not matrices of maximum rank. The case given by matrices of rank 1 was elaborated in the paper [31]. Now we consider another interesting case, where the functions P_i and Q_i take values in the space of $n \times n$ matrices of rank n_*. Such functions can be represented as[4]

$$P_i = u_i{}^t w_i, \qquad Q_i = x_i{}^t y_i, \tag{14}$$

where u, w, x and y are functions on $\mathcal{M}$ taking values in the space of $n \times n_*$ complex matrices of rank n_*. The used $\mathbb{Z}$-gradation suggests the most convenient block-matrix representation for the matrices of the form (14) as

$$(P_i)_{\alpha\beta} = u_{i,\alpha}{}^t w_{i,\beta}, \qquad (Q_i)_{\alpha\beta} = x_{i,\alpha}{}^t y_{i,\beta},$$

where the standard matrix multiplication of the $n_* \times n_*$ matrix-valued functions $u_{i,\alpha}$, $x_{i,\alpha}$ by the $n_* \times n_*$ matrix-valued functions ${}^t w_{i,\beta}$, ${}^t y_{i,\beta}$ is implied. Considering the rank of the matrices P_i and Q_i not to be equal to 1, we are looking for a novel generalization of the Abelian loop Toda soliton constructions [28] that would be different from those given in the paper [31].

Then, within the formalism of rational dressing we see that the functions w and x can be expressed via the functions y and u, and we find the expressions

$$(P_i)_{\alpha\beta} = -\frac{1}{p} u_{i,\alpha} \sum_{j=1}^{r} \left(R_\beta^{-1}\right)_{ij} {}^t y_{j,\beta}, \qquad (Q_i)_{\alpha\beta} = \frac{1}{p}\sum_{j=1}^{r} u_{j,\alpha} \frac{1}{\mu_j} \left(R_{\alpha+1}^{-1}\right)_{ji} \nu_i {}^t y_{i,\beta},$$

with $n_* r \times n_* r$ matrix-valued functions R_α defined through its $n_* \times n_*$ blocks as

$$(R_\alpha)_{ij} = \frac{1}{\nu_i^p - \mu_j^p} \sum_{\beta=1}^{p} \nu_i^{p-|\beta-\alpha|_p} \mu_j^{|\beta-\alpha|_p} {}^t y_{i,\beta} u_{j,\beta},$$

where $|x|_p$ denotes the residue of division of x by p. It is important to note here that, unlike the Abelian and non-Abelian cases considered in the papers [28, 29, 31], for any i, j the block $(R_\alpha)_{ij}$ is now an $n_* \times n_*$ matrix-valued function, that adds to the explicitly indicated summations over the pole indices respective matrix multiplications.

[4] Hereafter, the superscript t stands for the usual matrix transposition.

It is convenient to use quantities defined as $\widetilde{u}_{i,\alpha} = u_{i,\alpha}\mu_i^\alpha$, $\widetilde{y}_{i,\alpha} = y_{i,\alpha}\nu_i^{-\alpha}$ and $(\widetilde{R}_\alpha)_{ij} = \nu_i^{-\alpha}(R_\alpha)_{ij}\mu_j^\alpha$. For the matrices $\widetilde{R}_\alpha$ we have explicitly the relation

$$(\widetilde{R}_\alpha)_{ij} = \frac{1}{\nu_i^p - \mu_j^p}\left(\mu_j^p\sum_{\beta=1}^{\alpha-1}{}^t\widetilde{y}_{i,\beta}\widetilde{u}_{j,\beta} + \nu_i^p\sum_{\beta=\alpha}^{p}{}^t\widetilde{y}_{i,\beta}\widetilde{u}_{j,\beta}\right).$$

Then, in terms of these quantities, the $n_* \times n_*$ matrix-valued functions Γ_α can be written as

$$\Gamma_\alpha = I_{n_\alpha} - \sum_{i,j=1}^{r}\widetilde{u}_{i,\alpha}\left(\widetilde{R}_\alpha^{-1}\right)_{ij}{}^t\widetilde{y}_{j,\alpha}.$$

Similarly, for the corresponding inverse mappings we obtain the expression

$$\Gamma_\alpha^{-1} = I_{n_\alpha} + \sum_{i,j=1}^{r}\widetilde{u}_{i,\alpha}\left(\widetilde{R}_{\alpha+1}^{-1}\right)_{ij}{}^t\widetilde{y}_{j,\alpha},$$

which can be useful in verifying the Toda equations.

To finally satisfy the conditions imposed earlier on the matrix-valued functions P_i and Q_i, we also demand the validity of the equations

$$\partial_- u_i = -\mu_i^{-1}c_- u_i, \qquad \partial_+ u_i = -\mu_i c_+ u_i, \tag{15}$$

$$\partial_- y_i = \nu_i^{-1}{}^t c_- y_i, \qquad \partial_+ y_i = \nu_i{}^t c_+ y_i, \tag{16}$$

that are sufficient to fulfil relations (9), (10) and (11), (12). Using the explicit forms of the matrices $c_\pm$, we write the general solutions to (15), (16) as[5]

$$u_{i,\beta} = \sum_{\alpha=1}^{p}\epsilon_p^{\beta\alpha}\exp\left(-\mu_i^{-1}\epsilon_p^{-\alpha}z^- - \mu_i\epsilon_p^\alpha z^+\right)c_{i,\alpha}, \tag{17}$$

$$y_{i,\beta} = \sum_{\alpha=1}^{p}\epsilon_p^{\beta\alpha}\exp\left(\nu_i^{-1}\epsilon_p^{\alpha}z^- + \nu_i\epsilon_p^{-\alpha}z^+\right)d_{i,\alpha}, \tag{18}$$

where $c_{i,\alpha}$ and $d_{i,\alpha}$ are $n_* \times n_*$ complex matrices meaning the initial-value data for equations (15) and (16). With these solutions we immediately obtain for the blocks of the matrix-valued functions $\widetilde{R}_\alpha$ the following expression:

$$(\widetilde{R}_\alpha)_{ij} = \sum_{\beta,\delta=1}^{p}\mathrm{e}^{Z_{-\beta}(\nu_i)-Z_\delta(\mu_j)}\frac{\epsilon_p^{\alpha(\beta+\delta)}}{1-\mu_j\nu_i^{-1}\epsilon_p^{\beta+\delta}}\nu_i^{-\alpha}({}^t d_{i,\beta}c_{j,\delta})\mu_j^\alpha,$$

where we have introduced the notation $Z_\alpha(\mu_i) = \mu_i^{-1}\epsilon_p^{-\alpha}z^- + \mu_i\epsilon_p^\alpha z^+$.

3. Soliton-like solutions

To construct solutions making sense as r-solitons, that is, by the definition we use here, solutions depending on r linear combinations of independent variables z^+ and z^-, we assume that for each value of the index $i = 1, \dots, r$ the initial-value data of the Toda system under consideration are such that matrix-valued coefficients $c_{i,\alpha}$ are different from zero for only one value of α, which we denote by I_i, and that the matrix-valued coefficients $d_{i,\alpha}$ are different

[5] It should be instructive to confer these expressions with those derived in the paper [31] for the case of general $\mathbb{Z}$-gradations of inner type.

from zero for only two values of α, which we denote by J_i and K_i. We also use for such non-vanishing initial-data $n_* \times n_*$ matrices the notation $d_{J_i} = d_{i,J_i}$, $d_{K_i} = d_{i,K_i}$ and $c_{I_i} = c_{i,I_i}$. For the $n_* \times n_*$ blocks of the matrix-valued functions $\widetilde{u}_i$ and $\widetilde{y}_i$ this assumption gives

$$\widetilde{u}_{i,\alpha} = \mu_i^{\alpha} \epsilon_p^{\alpha I_i} \,\mathrm{e}^{-Z_{I_i}(\mu_i)} c_{I_i}, \tag{19}$$

$${}^t\widetilde{y}_{i,\alpha} = \nu_i^{-\alpha} \epsilon_p^{\alpha J_i} \,\mathrm{e}^{Z_{-J_i}(\nu_i)t} d_{J_i} + \nu_i^{-\alpha} \epsilon_p^{\alpha K_i} \,\mathrm{e}^{Z_{-K_i}(\nu_i)t} d_{K_i}. \tag{20}$$

With these relations, we can write the expression for the mappings Γ_α in the form

$$\Gamma_\alpha = I_{n_\alpha} - \sum_{i,j=1}^{r} c_{I_i} \big(\widetilde{R}_\alpha^{\prime -1}\big)_{ij} \big({}^t d_{J_j} + E_{\alpha,j}\, {}^t d_{K_j}\big), \tag{21}$$

where we have used the notation

$$(\widetilde{R}'_\alpha)_{ij} = \widetilde{D}_{ij}(J) + E_{\alpha,i} \widetilde{D}_{ij}(K), \qquad E_{\alpha,i} = \epsilon_p^{\alpha\rho_i} \,\mathrm{e}^{Z_i(\zeta)}, \tag{22}$$

with the dependence on the variables z^+ and z^- given through the functions

$$Z_i(\zeta) = \kappa_{\rho_i} \big(\zeta_i^{-1} z^- + \zeta_i z^+\big),$$

and the convenient parameters $\rho_i = K_i - J_i$, $\zeta_i = -\mathrm{i}\nu_i \epsilon_p^{-(K_i+J_i)/2}$, $\kappa_{\rho_i} = 2\sin(\pi\rho/p)$, and besides

$$\widetilde{D}_{ij}(A) = \frac{{}^t d_{A_i} c_{I_j}}{1 - \nu_i^{-1} \mu_j \epsilon_p^{A_i + I_j}}, \qquad i, j = 1, \ldots, r, \tag{23}$$

for the $n_* \times n_*$ blocks of the $n_* r \times n_* r$ matrices $\widetilde{D}(A)$, $A = J, K$. Similar $r \times r$ matrices were introduced already in our previous paper [31] for the rank-1 case, however now, in the case of rank-n_*, we have a different situation, such that $\widetilde{D}_{ij}$ itself is a complex $n_* \times n_*$ matrix for each i and j.

We assume that the $n_* \times n_*$ matrices c_{I_i} are non-degenerate. Let us multiply Γ_α in (21) by c_{I_ℓ} from the right-hand side and write

$$\Gamma_\alpha c_{I_\ell} = \sum_{i,j=1}^{r} c_{I_i} \big(\widetilde{R}_\alpha^{\prime -1}\big)_{ij} \big[(\widetilde{R}'_\alpha)_{j\ell} - \big({}^t d_{J_j} + E_{\alpha,j}\, {}^t d_{K_j}\big) c_{I_\ell}\big]. \tag{24}$$

It is not difficult to see that

$$(\widetilde{R}'_\alpha)_{j\ell} - \big({}^t d_{J_j} + E_{\alpha,j}\, {}^t d_{K_j}\big) c_{I_\ell} = \nu_j^{-1} \epsilon_p^{J_j} (\widetilde{R}'_{\alpha+1})_{j\ell} \mu_\ell \epsilon_p^{I_\ell}.$$

Now, multiplying (24) from the right-hand side by the inverse matrix $c_{I_\ell}^{-1}$ and summing up over $\ell = 1, \ldots, r$, we obtain the following expression:

$$\Gamma_\alpha = \frac{1}{r} \sum_{i,j,k=1}^{r} c_{I_i} \big(\widetilde{R}_\alpha^{\prime -1}\big)_{ij} \nu_j^{-1} \epsilon_p^{J_j} (\widetilde{R}'_{\alpha+1})_{jk} \mu_k \epsilon_p^{I_k} c_{I_k}^{-1}.$$

To give the soliton solutions a final form, it is also convenient to make use of simplest symmetries of the Toda equation. It is clear from relations (3) that the transformations

$$\Gamma_\alpha \to \xi \Gamma_\alpha, \qquad \Gamma_\alpha \to x^{-1} \Gamma_\alpha x \tag{25}$$

for a nonzero constant ξ and a non-singular constant $n_* \times n_*$ matrix x are symmetry transformations of the Toda system under consideration.

In particular, using the symmetry transformations (25) with $\xi = \mu^{-1} \nu \epsilon_p^{-(I+J)}$ and $x = c_I$, we can write the one-soliton solution as

$$\Gamma_\alpha = \widetilde{R}_\alpha^{\prime -1} \widetilde{R}'_{\alpha+1}, \tag{26}$$

where the matrices $\widetilde{R}'_\alpha$ are explicitly given by relations (22) and (23). It reproduces exactly the one-soliton solution constructed in the paper [30] by means of a different approach. The solution (26) can also be written in a convenient form

$$\Gamma_\alpha = T_\alpha^{-1} T_{\alpha+1},$$

where $T_\alpha = I_{n_\alpha} + E_\alpha H$ and $H = \widetilde{D}(K)\widetilde{D}^{-1}(J)$.

Also the multi-soliton solutions, $r \geqslant 2$, can be written in a compact form,

$$\Gamma_\alpha = {}^T c_I \big(\widetilde{R}'^{-1}_\alpha\big) \mathcal{N}_J^{-1} (\widetilde{R}'_{\alpha+1}) \mathcal{M}_I c_I^{-1}. \qquad (27)$$

Here we use the notation ${}^T c_I$ and c_I^{-1} for $n_* \times n_* r$ and $n_* r \times n_*$ matrices, respectively, defined as ${}^T c_I = \big(c_{I_1} \dots c_{I_r}\big)$ and ${}^t c_I^{-1} = \big({}^t c_{I_1}^{-1} \dots {}^t c_{I_r}^{-1}\big)$, and the notation $\mathcal{N}_J^{-1}$ and $\mathcal{M}_I$ for block-diagonal $n_* r \times n_* r$ matrices defined as

$$\mathcal{N}_J^{-1} = \begin{pmatrix} \nu_1^{-1} \epsilon_p^{J_1} I_{n_*} & & \\ & \ddots & \\ & & \nu_r^{-1} \epsilon_p^{J_r} I_{n_*} \end{pmatrix}, \qquad \mathcal{M}_I = \begin{pmatrix} \mu_1 \epsilon_p^{I_1} I_{n_*} & & \\ & \ddots & \\ & & \mu_r \epsilon_p^{I_r} I_{n_*} \end{pmatrix}.$$

Putting $n_* = 1$ we can recover the Abelian case analyzed in the paper [28]. The solutions (26) and (27) may be regarded as a novel non-Abelian generalization, complementary to that of the paper [31], of Hirota's soliton constructions [15] successfully used for the Abelian loop Toda systems. Recall that the τ-functions of the Abelian construction of Hirota, the τ_α were generalized by the set of matrix-valued functions $\widetilde{T}_\alpha^X$ and ordinary functions $\widetilde{T}_\alpha$ in the non-Abelian case [31], while now, in a higher-rank case, we have the set of matrix-valued functions $\widetilde{R}'_{\alpha+1} \mathcal{M}_I c_I^{-1}$ and ${}^T c_I \widetilde{R}'^{-1}_\alpha \mathcal{N}_J^{-1}$, taken always in a respective combination, instead of the functions τ_α.

To obtain more solutions to these equations, not necessarily soliton-like, one should keep nonzero more initial-value data $c_{i,\alpha}$ and $d_{i,\alpha}$ entering (17) and (18), that leads to more general expressions for $\widetilde{u}_{i,\alpha}$ and $\widetilde{y}_{i,\alpha}$ than (19) and (20).

4. Discussion

It should be rather illuminating to reproduce our results along the lines of any other approaches. One such possibility would be, probably, to try the general dressing procedure [25, 26, 34]. Here, after the dressing transformations are completed, the problem is reduced to certain spectral problems for the matrices c_+ and c_-. Thus, using specific vertex operators V_i related to an appropriate basis, one could bring the generalized τ-functions to the form

$$\prod_{i=1}^{r} \big(I_{n_\alpha} + E_{\alpha,i} V_i + \cdots\big),$$

where the vertex operators are supposed to obey certain nilpotency conditions [25, 26]. As we mentioned at the beginning, the resulting expressions can be then directly compared to what one finds in the case of Abelian Toda systems. However, it is not clear yet how to provide the corresponding statement for the non-Abelian case, while preserving the block-matrix structure suggested by the $\mathbb{Z}$-gradation.

The next step, most naturally following our constructions, would be a thorough investigation of the physical content of the solutions. In this way, one should describe standard properties defining the solitons, for example, in the spirit of the paper [16]. Note that such a program, in general, must be based on the specification of real forms of the loop Toda equations under consideration, so that the solutions making sense as 'physical solitons' could be found in the present ones by certain reductions. Here, such reductions impose certain conditions on the characteristic parameters entering the explicit forms of the soliton-like solutions. In particular, it can be shown that to compact real forms specified by the conditions $\Gamma_\alpha^\dagger = \Gamma_\alpha^{-1}$, where dagger means the Hermitian conjugation, actually correspond such 'physical solitons', and there are no such solutions in the case of non-compact real forms corresponding to the condition $\Gamma_\alpha^* = \Gamma_\alpha$, where star denotes the usual complex conjugation.

A few comments are in order. To have a solution making sense as a soliton, we must require that $\mu_i^* = \nu_i$. Putting $r = 1$ we see that it should be valid $H^\dagger \equiv (H' \exp\delta)^\dagger = \epsilon_p^\rho H' \exp\delta$, where we have used the notation $\exp\delta = \big(1 - \mu\nu^{-1}\epsilon_p^{I+J}\big)\big/\big(1 - \mu\nu^{-1}\epsilon_p^{I+K}\big)$. Note that the function $Z(\zeta)$ should be real, and so, if $z^- = x - \mathrm{i}t$, $z^+ = x + \mathrm{i}t$, this function can be written in a familiar form, as $2\kappa_\rho(x - vt)/\sqrt{1+v^2}$, where v is the ratio of the imaginary and real parts of the parameter ζ, $|\zeta|^2 = 1$, and if $z^- = x - t$, $z^+ = x + t$, as $2\kappa_\rho(x + vt)/\sqrt{1-v^2}$, where $v = (\zeta - 1/\zeta)/(\zeta + 1/\zeta)$ for a real ζ. In the Abelian limit, when $n_* = 1$, the matrix H' is just a complex number, and it can be lifted up to $\exp\delta$, and then the above relation, that is a restriction on the initial-value data, fixes the imaginary part of total δ to be $-\pi\rho/p$.

We would like to refer to the paper [11], where the above real forms of loop Toda equations were obtained for the case of $p = 2$, and to the paper [12], where basic physical properties of one-soliton and two-soliton (soliton—anti-soliton and breather) solutions were investigated for a matrix generalization of the sine-Gordon equation based on the coset $\mathrm{SU}_2 \times \mathrm{SU}_2/\mathrm{SU}_2$. In the Abelian case the physical properties of loop Toda solitons, including masses, topological charges, scattering processes, were described in the paper [16]. We will address these issues about our non-Abelian constructions in forthcoming publications.

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References

[1] Leznov A N and Saveliev M V 1992 *Group-Theoretical Methods for the Integration of Nonlinear Dynamical Systems* (Basel: Birkhauser)

[2] Razumov A V and Saveliev M V 1997 *Lie Algebras, Geometry and Toda-Type Systems* (Cambridge: Cambridge University Press)

[3] Mikhailov A V 1979 Integrability of a two-dimensional generalization of the Toda chain *Sov. Phys.—JETP Lett.* **30** 414–8

[4] Mikhailov A V 1981 The reduction problem and the inverse scattering method *Physica* **3D** 73–117

[5] Mikhailov A V, Olshanetsky M A and Perelomov A M 1981 Two-dimensional generalized Toda lattice *Commun. Math. Phys.* **79** 473–88

[6] Khastgir S P and Sasaki R 1996 Non-canonical folding of Dynkin diagrams and reduction of affine Toda theories *Prog. Theor. Phys.* **95** 503–18

[7] Olive D I and Turok N 1986 The Toda lattice field theory hierarchies and zero-curvature conditions in Kac–moody algebras *Nucl. Phys.* B **265** 469–84

[8] Constantinidis C P, Ferreira L A, Gomes J F and Zimerman A H 1993 Connection between affine and conformal affine Toda models and their Hirota's solution *Phys. Lett.* B **298** 88–94 (arXiv:hep-th/9207061)

[9] Nirov Kh S and Razumov A V 2006 On $\mathbb{Z}$-gradations of twisted loop Lie algebras of complex simple Lie algebras *Commun. Math. Phys.* **267** 587–610 (arXiv:math-ph/0504038)

[10] Nirov Kh S and Razumov A V 2007 Toda equations associated with loop groups of complex classical Lie groups *Nucl. Phys.* B **782** 241–75 (arXiv:math-ph/0612054)

[11] Nirov Kh S and Razumov A V 2008 $\mathbb{Z}$-graded loop Lie algebras, loop groups, and Toda equations *Theor. Math. Phys.* **154** 385–404 (arXiv:0705.2681)

[12] Park Q-Han and Shin H J 1996 Classical matrix sine-Gordon theory *Nucl. Phys.* B **458** 327–54 (arXiv:hep-th/9505017)

[13] Fernández-Pousa C R, Gallas M V, Hollowood T J and Miramontes J L 1997 The symmetric space and homogeneous sine-Gordon theories *Nucl. Phys.* B **484** 609–30 (arXiv:hep-th/9606032)

[14] Miramontes L 1999 Tau-functions generating the conservation laws for generalized integrable hierarchies of KdV and affine-toda type *Nucl. Phys.* B **547** 623–63 (arXiv:hep-th/9809052)

[15] Hirota R 2004 *The Direct Method in Soliton Theory* (Cambridge: Cambridge University Press)

[16] Hollowood T 1992 Solitons in affine Toda field theories *Nucl. Phys.* B **384** 523–40

[17] MacKay N J and McGhee W A 1993 Affine Toda solutions and automorphisms of Dynkin diagrams *Int. J. Mod. Phys.* A **8** 2791–807 (arXiv:hep-th/9208057)

MacKay N J and McGhee W A 1993 Affine Toda solutions and automorphisms of Dynkin diagrams *Int. J. Mod. Phys.* A **8** 3830 (erratum)

[18] Aratyn H, Constantinidis C P, Ferreira L A, Gomes J F and Zimerman A H 1993 Hirota's solitons in the affine and the conformal affine Toda models *Nucl. Phys.* B **406** 727–70 (arXiv:hep-th/9212086)

[19] Zhu Z and Caldi D G 1995 Multi-soliton solutions of affine Toda models *Nucl. Phys.* B **436** 659–80 (arXiv:hep-th/9307175)

[20] Khastgir S P and Sasaki R 1996 Instability of solitons in imaginary coupling affine Toda field theory *Prog. Theor. Phys.* **95** 485–502

[21] Olive D I, Turok N and Underwood W R 1993 Solitons and the energy-momentum tensor for affine Toda theory *Nucl. Phys.* B **401** 663–97

[22] Olive D I, Turok N and Underwood W R 1993 Affine Toda solitons and vertex operators *Nucl. Phys.* B **409** 509–46

[23] Kneipp M A C and Olive D I 1996 Solitons and vertex operators in twisted affine Toda field theories *Commun. Math. Phys.* **177** 561–82

[24] Olive D I, Saveliev M V and Underwood J W R 1993 On a solitonic specialisation for the general solutions of some two-dimensional completely integrable systems *Phys. Lett.* B **311** 117–22 (arXiv:hep-th/9212123)

[25] Ferreira L A, Miramontes J L and Guillén J S 1995 Solitons, τ-functions and Hamiltonian reduction for non-Abelian conformal affine Toda theories *Nucl. Phys.* B **449** 631–79 (arXiv:hep-th/9412127)

[26] Ferreira L A, Miramontes J L and Guillén J S 1997 Tau-functions and dressing transformations for zero-curvature affine integrable equations *J. Math. Phys.* **38** 882–901 (arXiv:hep-th/9606066)

[27] Zakharov V E and Shabat A B 1979 Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II *Func. Anal.* **13** 166–75

[28] Nirov Kh S and Razumov A V 2008 Abelian Toda solitons revisited *Rev. Math. Phys.* **20** 1209–48 (arXiv:0802.0593)

[29] Nirov Kh S and Razumov A V 2008 The rational dressing for Abelian twisted loop Toda systems *J. High Energy Phys.* JHEP12(2008)048 (arXiv:0806.2597)

[30] Etingof P, Gelfand I and Retakh V 1998 Non-Abelian integrable systems, quasideterminants, and Marchenko Lemma *Math. Res. Lett.* **5** 1–12 (arXiv:q-alg/9707017)

[31] Nirov Kh S and Razumov A V 2009 Solving non-Abelian loop Toda equations *Nucl. Phys.* B (doi:10.1016/j.nuclphysb.2009.01.010) (arXiv:0809.3944)

[32] Kriegl A and Michor P 1991 Aspects of the theory of infinite dimensional manifolds *Diff. Geom. Appl.* **1** 159–76 (arXiv:math.DG/9202206)

[33] Kriegl A and Michor P 1997 *The Convenient Setting of Global Analysis (Mathematical Surveys and Monographs* vol 53*)* (Providence, RI: American Mathematical Society)

[34] Babelon O and Bernard D 1992 Dressing symmetries *Commun. Math. Phys.* **149** 279–306